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LETTER TO THE EDITOR

Universal amplitude in the sizes of rings in two dimensions

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Abstract. For self-avoiding rings of N steps in two dimensions, the limiting value as $N \to \infty$ of the combination $Np_N \langle R^2 \rangle_N x_c^N$ (where p_N is the number of distinct rings, $\langle R^2 \rangle_N$ is their mean square radius of gyration, and x_c is the critical fugacity) is equal to a calculable lattice-dependent number times a universal amplitude. This latter quantity is calculated exactly using methods of conformal invariance. The value is in good agreement with the results of enumeration studies.

Recently a new result has been obtained which extends the quantitative predictions of conformal invariance in two-dimensional critical phenomena into the scaling region, away from the critical point (Cardy 1988). It is a quantitative extension of the 'c-theorem' of Zamolodchikov (1986), and it relates the value of the conformal anomaly number c, which characterises the critical point theory, to the second moment of the energy–energy correlations in the non-critical theory:

$$c = 3\pi t^2 (2 - x_E)^2 \int r^2 \langle E(r) E(0) \rangle_c \, d^2 r \tag{1}$$

where t is proportional to $T - T_c$, $E(r)$ is the energy density, and $x_E = 2 - \nu^{-1}$ is its scaling dimension. The combination tE is normalised so that the reduced Hamiltonian is $\mathcal{H}_c + t \int E(r) \, d^2 r$, in the continuum limit. Equation (1) has been checked for the Ising model (Cardy 1988), where the energy correlations are known exactly (Hecht 1967). As explained in Cardy (1988), equation (1) is equivalent to a prediction for the universal quantity $f_s \xi^2$, where f_s is the singular part of the free energy per unit area, and the correlation length ξ is defined, at least for $\alpha > 0$, in terms of the second moment of the energy correlations.

In this letter, we apply this formula to the limit $n \to 0$ of the O(n) model, which corresponds to self-avoiding walks (de Gennes 1972, des Cloizeaux 1975). In this mapping, the magnetic correlations give information about open walks, or linear polymers, while the energy correlations relate to closed walks, or rings. It turns out that the universality of the right-hand side of (1), or, equivalently of $f_s \xi^2$, implies the universality (up to calculable lattice-dependent factors) of the combination $Np_N \langle R^2 \rangle_N x_c^N$ defined in the abstract. In addition, from the known dependence of $c(n)$ for the O(n) model, we can compute its value. The universality of this combination, with, however, $\langle R^2 \rangle_N$ being the mean square radius of gyration of *open* N-step walks, was first predicted, and its lattice independence verified, by Privman and Redner (1985).

Explicitly, consider a regular lattice with a reduced Hamiltonian

$$\mathcal{H} = -x \sum_{\text{bonds}} E(r). \tag{2}$$

Here $E(r) = s_i \cdot s_j$, where i, j label the sites at the end of the bond at r, and the s_i are n-component vectors, normalised so that $\mathrm{Tr}\, s_i^a s_j^b = \delta^{ab}\delta_{ij}$. This theory has a critical point at $x = x_c$. Equation (1) may then be written on the lattice as a double sum over bonds r, r':

$$c(n) = 3\pi(x_c - x)^2 A^{-1}\left(\frac{A}{N_b}\right)^2 \sum_{r,r'} (r - r')^2 \langle E(r)E(r')\rangle_c \tag{3}$$

where A is the total area and N_b is the total number of bonds in the lattice. If the correlation function is evaluated in an expansion in x, it is given, as $n \to 0$, by a sum of diagrams in which two self-avoiding walks connect the ends of the bonds at r, r' to form a single ring. The contribution of a given ring of length N, with bonds at $(r_1, r_2, \ldots)$, to the sum in (2) is then

$$2nx^{N-2} \sum_{k,l} (r_k - r_l)^2. \tag{4}$$

Note the factor of two, which corresponds to the fact that, in the sum over pairs of walks, each ring appears twice. The sum in (4) is, for a given ring, equal to $2N^2 R_b^2$, where R_b is its bond-weighted radius of gyration, that is, calculated by imagining that equal masses are situated at the midpoints of the bonds. If the average of this quantity over all rings of length N is denoted by $\langle R_b^2\rangle_N$ and the total number of such rings is $N_s p_N$, where N_s is the total number of sites in the lattice, then the sum in (3) may be written as

$$4nN_s \sum_N N^2 p_N \langle R_b^2\rangle_N x^{N-2} \tag{5}$$

so that finally, taking the limit $n \to 0$,

$$\sum_N N^2 p_N \langle R_b^2\rangle_N x^N = \left(\frac{N_b}{A}\right)^2 \left(\frac{N_s}{A}\right)^{-1} \frac{c'(0)}{12\pi(2 - x_E)^2} \frac{x^2}{(x_c - x)^2}. \tag{6}$$

The value of the central charge $c(n)$ for the O(n) model is known, both from matching exponents (Dotsenko and Fateev 1984, Singh and Shastry 1985), and from calculations of the free energy of a frustrated Gaussian model defined on a cylinder (Blöte *et al* 1986) to be $c = 1 - 6/m(m+1)$ where $n = 2\cos(\pi/m)$. Note that c vanishes at $n = 0$, as it must, since the free energy is then zero. However, its derivative is finite, and we find $c'(0) = 5/3\pi$. The fact that the right-hand side of (6) behaves like $(x - x_c)^{-2}$ as $x \to x_c$ implies that $N^2 p_N \langle R_b^2\rangle_N$ behaves like $N x_c^{-N}$ as $N \to \infty$. Using the result $2 - x_E = \frac{4}{3}$, we then find that for a square lattice

$$\lim_{N\to\infty} N p_N \langle R_b^2\rangle_N x_c^N = \frac{5}{16\pi^2}. \tag{7}$$

Privman and Rudnick (1985) have measured the radii of gyration of rings on the square lattice up to 28 steps. They in fact consider the site-weighted radius of gyration R_s, but it is straightforward to show that this is related to the bond-weighted case by $R_b^2 = R_s^2 - \frac{1}{4}$. In comparing the prediction of equation (6) with this data we have chosen to treat the factor of x^2 in the numerator exactly, not replacing it by x_c^2. This corresponds to including a correction factor $(1 - 2/N)^{-1}$ on the left-hand side of equation (7) for finite N. This procedure eliminates the strongest of the $O(N^{-1})$ corrections. In figure 1 we show the results for this quantity, using the central value for the critical fugacity $x_c^{-1} = 2.638\,155$ obtained from an enumeration of rings up to 46 steps by Enting and

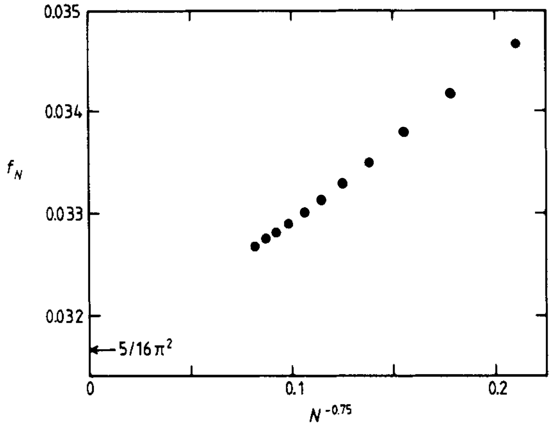

Figure 1. The quantity $f_N = (1-2/N)^{-1} N p_N \langle R_b^2 \rangle_N x_c^N$ against $N^{-0.75}$. The predicted asymptotic value of $5/16\pi^2$ is shown.

Guttmann (1985). Anticipating a correction to the scaling term of the form $N^{-\theta}$, the data are plotted against this quantity, using the theoretically expected value $\theta = \nu = 0.75$ (Privman *et al* 1984). As may be seen, they extrapolate well to the predicted asymptotic value, although some curvature appears in the plot for smaller values of N. A better straight line fit may be obtained by taking an effective exponent $\theta = 0.85$, without disturbing the agreement with the asymptotic value.

To summarise, we have made a prediction of conformal invariance for the sizes of self-avoiding rings, and shown that it agrees well with numerical data. Since a number of different elements of the theory went into this prediction, its success may be viewed as an important confirmation of these ideas.

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